Complex Numbers

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Complex Numbers - The equation $x^2=-1$ has no solution in the set of real numbers because the square of every real number is either positive or zero.

Definition A number of the form $x+iy$, where $i=\sqrt{(-1)}$ $[x^2=-1 \Rightarrow x=\sqrt{-1}=i]$ and x, y are both real numbers, is called a complex number. A complex number $x+iy$ or (x, y) is usually denoted by the symbol z. If we write $z=x+iy$ then x is called the real part and y the imaginary part of the complex number z and these are denoted by $R(z)$ and $I(z)$ respectively. Thus in the complex number $z=\sqrt{3}+5i$

$R(z)$ = the real part of $z=\sqrt{3}$,

$I(z)$ = the imaginary part of $z = 5$

$\Rightarrow$ A complex number is said to be purely real if its imaginary part is zero, and purely imaginary if its real part is zero.

Equality of two complex numbers - Two complex numbers are equal if and only if the real part of one is equal to the real part of the other and the imaginary part of one is equal to the imaginary part of the another.

Ex. $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

If $z_1=z_2$ then
$x_1=x_2$ and $y_1=y_2$

Addition of Complex Numbers

if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two Complex numbers then the sum of z_1 and z_2

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$= \underbrace{(x_1 + x_2)}_{\text{both are real parts of } z_1+z_2} + i\underbrace{(y_1 + y_2)}_{\text{both are imaginary parts of } z_1+z_2}$$

Ex. If $z_1 = 2 + 3i$, $z_2 = 7 + 8i$

$$\therefore z_1 + z_2 = 2 + 3i + 7 + 8i$$

$$= 9 + 10i$$

Multiplication of Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are any two complex numbers, then product of z_1 and z_2 is

$$z_1 . z_2 = (x_1 + iy_1)(x_2 + iy_2) \quad [\because i^2 = -1$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Difference of Two Complex Numbers

If z_1 and z_2 are two complex numbers then

$z_1 - z_2 = z_1 + (-z_2)$ is difference of two complex numbers.

Ex $z_1 = 5 + 3i$, $z_2 = 1 + i$

$$z_1 - z_2 = (5 + 3i) - (1 + i)$$

$$= 4 + 2i$$

Division in Complex Numbers

A complex number $a+ib$, is said to be divisible by a complex number $c+id$ if there exists a complex number $x+iy$ such that

$$(x+iy) = \frac{a+ib}{c+id}$$

$$= \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} \quad [\because i^2=-1$$

$$\therefore x = \frac{ac+bd}{c^2+d^2}, \quad y = \frac{bc-ad}{c^2+d^2}$$

provided $c^2+d^2 \neq 0$ which implies that c and d are not both zero.

The Symbol i and its powers

$i^2 = -1$, then

$i^3 = -i$, $i^4 = 1$, $i^5 = i$

$i^6 = -1$, and so on.

Conjugate of a Complex Number

if $z = x+iy$ is any complex number, then the complex number $x-iy$ is called the conjugate of the complex number z and is written as $\bar{z}$.

~~this~~ if $z = 2+3i$ then $\bar{z} = 2-3i$

The following results are obvious and should be remembered.

(i) Two complex numbers are equal if and only if their conjugates are equal i.e.

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$z_1 = z_2$ if and only if $\bar{z}_1 = \bar{z}_2$

(ii) $\overline{(\bar{z})} = z$

(iii) we have $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, provided $z_2 \neq 0$

(iv) if $z = x + iy$ then

$z + \bar{z} = x + iy + x - iy = 2x$
$= 2R(z)$

(v) A complex number $z = x + iy$ is purely imaginary if and only if $z + \bar{z} = 0$

(vi) if $z = x + iy$, then $z - \bar{z} = x + iy - (x - iy)$
$= 2iy$
$= 2iI(z)$

(vii) A complex number z is purely real if and only if $z - \bar{z} = 0$

(viii) if $z = x + iy$, then $z\bar{z} = (x + iy)(x - iy)$
$= x^2 + y^2$

Modulus of a Complex Number

if $z = x + iy$ be any complex number, then the non-negative real number $\sqrt{(x^2 + y^2)}$ is called the modulus or absolute value of the complex number z and is denoted by $|z|$ or mod z.

Ex $z = 3 + 4i$

$|z| = \sqrt{(3^2 + 4^2)} = \sqrt{9 + 16}$

$|z| = 5$

$\Rightarrow$ The modulus of a complex number is equal to the positive square root of the sum of the squares of the real and imaginary parts of that complex number.

The following result about the modulus of a complex number should be remembered.

(i) If z is any complex number, then $|z| = |\bar{z}|$

If $z = x + iy$ then $|z| = \sqrt{(x^2+y^2)}$

Also $|\bar{z}| = |x - iy| = \sqrt{\{x^2 + (-y)^2\}} = \sqrt{x^2+y^2} = |z|$

(ii) if $z = x + iy$ be any complex number, then

$$|z| = 0 \Leftrightarrow \sqrt{x^2+y^2} = 0 \Leftrightarrow x^2+y^2 = 0$$

$$\Rightarrow x = 0, \ y = 0$$

$$\underline{z = 0 + i0 = 0}$$

(iii) if z is any complex number, then $z\bar{z} = |z|^2$

$\Rightarrow z = x + iy$ then

$$z\bar{z} = (x+iy)(x-iy) = x^2+y^2$$

$$= \left(\sqrt{(x^2+y^2)}\right)^2 = |z|^2$$

(iv) If z_1, z_2 are any two complex numbers then

$$|z_1 z_2| = |z_1||z_2|$$

i.e the modulus of a product is equal to the product of the moduli.

(v) If z_1, z_2 are any two complex numbers and $z_2 \neq 0$ then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

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Modulus - Argument form or Polar Standard form or Trigonometric form of a Complex Number

Every non-zero complex $x+iy$ can always be put in the form $r(\cos\theta + i\sin\theta)$, where r and θ both are real numbers.

let $x+iy = r(\cos\theta + i\sin\theta)$
$= r\cos\theta + ir\sin\theta$

Now equating real and imaginary parts on both side then we find

$$x = r\cos\theta \text{ ——— } (1)$$
$$y = r\sin\theta \text{ ——— } (2)$$

then squaring and adding equation (1) & (2)

$$x^2+y^2 = r^2$$

$\Rightarrow \quad r = \pm\sqrt{x^2+y^2}$ taking the positive sign

$\Rightarrow \quad r = |z|$

Thus r is known and is equal to the modulus of the complex number z.

Now substituting this value of r in equation (1) & (2)

$$\cos\theta = \frac{x}{\sqrt{(x^2+y^2)}}, \quad \sin\theta = \frac{y}{\sqrt{(x^2+y^2)}} \text{ ——— } (3)$$

Whatever be the values of x and y, if they are not both zero, there is one only one value of θ lying between $-\pi$ and π which satisfies the equation (3).

Thus θ is also determined. if n is any integer, then

$$\cos(2n\pi+\theta) = \cos\theta \text{ and } \sin(2n\pi+\theta) = \sin\theta$$

So there are an infinite numbers of values of θ satisfying the equation (3). Any value of θ satisfying the equation ③ is called an argument or amplitude of the complex number z

$$\theta = \arg z \text{ or } \operatorname{amp} z$$

Thus every non-zero complex number $x+iy$ can always be put uniquely in the form $r(\cos\theta + i\sin\theta)$, where r is positive and $-\pi < \theta < \pi$. This trigono-metical form of a complex number is also called its polor standard form or modulus-amplitude form.

Here we have seen that argument of a complex number is not unique. Thus arg z is many-valued function. The value of argument which satisfies the inequality $-\pi < \theta < \pi$ is called the Principal value of the argument and it is unique. If θ is the Principal value of arg z, then $2n\pi + \theta$, where n is any integer, is the general value of arg z and it represented by Arg z

thus $\boxed{\operatorname{Arg} z = 2n\pi + \arg z}$

De Moivre's Theorem

For all values of n, integral or fractional, positive or negative, the value or one of the value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$

or

$(\cos\theta - i\sin\theta)^n$ is $\cos n\theta - i\sin n\theta$