

# Special Theory of Relativity

*For*

**B.Sc. (III) Year (Physics)**

Prepared By

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**Special Theory of Relativity (STR):** Study of motion of a body in the inertial frame of references. It was given by *Albert Einstein* in 1905.

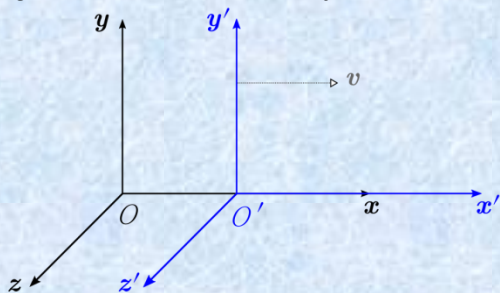
### Two postulates of Special Theory of Relativity:

1. All laws of physics are same for the observers in all inertial frame of references
2. The speed of light in a vacuum (  $c = 3 \times 10^8$  m/s ) is the same for all observers, regardless of their relative motion or of the motion of the source of the light

**Frame of reference:** The well defined coordinate system with respect to which the motion of any object is defined.

### Types of frame of references:

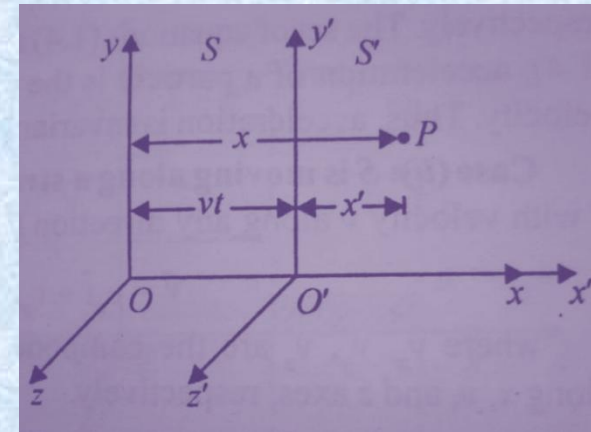
- Inertial or un-accelerated frame of reference:** The frame of reference in which objects obey Newton's law of inertia and other laws of Newtonian mechanics.
- Non-inertial or accelerated frame of reference:** The frame of reference in which objects do not obey Newton's laws.



This fig. Shows two inertial frame of references one is stationary and other is moving with some constant velocity  $v$ .

**Galilean Transformation:** Transforming the coordinates of one inertial frame to another.

Consider two inertial frame of references  $S$  and  $S'$  as shown in the figure.  $S$  is the coordinate system of  $(x, y, z, t)$  and  $S'$  is the coordinate system of  $(x', y', z', t')$ . Let the  $S'$  is moving with some velocity  $v$  with respect to (w.r.t.)  $S$  in +ve direction of  $x$ -axis.



In STR, we take coordinates in both space  $(x, y, z)$  and time  $t$ . Therefore position of any point will be taken as  $(x, y, z, t)$

At time  $t=0$  both the frame of reference coincide.

If any event happens at point  $P$  at any time  $t$  as shown in fig. and observations are taken by both the observers sitting at point  $O$  and  $O'$  in  $S$  and  $S'$  frame of references. Then we have

$$\left. \begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \right\} \quad (1)$$

If we assume  $S'$  is at rest and  $S$  is moving with velocity  $v$  in negative direction of  $x$ -axis then

$$\left. \begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned} \right\} \quad (2)$$

**NOTE:** Set of equations (1) is called Direct Galilean Transformation and set of equations (2) is called Inverse Galilean Transformation

**Lorentz Transformation:** Galilean transformations are valid for low speed  $v$  of moving frame of reference. Lorentz replaced the Galilean transformations and produce more accurate results for both low and high  $v$ .

Consider two inertial frames  $S$  and  $S'$ .  $S'$  is moving with constant velocity  $v$  w.r.t.  $S$  in +ve x-axis.

Lorentz assumed the relationship between  $x$  and  $x'$  as

$$x' = \lambda(x - vt) \quad (1) \quad \text{where, } \lambda \text{ is any undetermined constant}$$

According to First postulate of STR

$$x = \lambda(x' + vt') \quad (2) \quad \text{Moreover, } y' = y \text{ and } z' = z$$

From (1) and (2)

$$t' = \lambda t + \left( \frac{1 - \lambda^2}{\lambda v} \right) x \quad (3)$$

Consider a light signal produced at  $t = t' = 0$  when the origins of two frames coincide. This signal propagates as time passes.

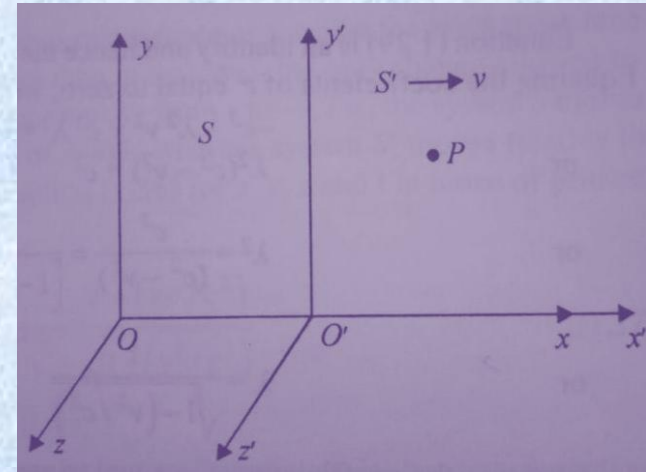
Now, apply second postulate of STR (The velocity of light  $c$  is same in  $S$  and  $S'$ )

After time  $t$  the distance covered by light pulse as observed in  $S$        $x = ct$       (4)

After time  $t'$  the distance covered by light pulse as observed in  $S'$        $x' = ct'$       (5)

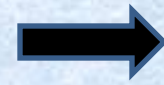
Substitute the values of  $x'$  and  $t'$  from equations (5) and (3) into equation (2) and after solving we get

$$x = ct \left[ \frac{1 + \frac{v}{c}}{1 - \left( \frac{1}{\lambda^2} - 1 \right) \frac{c}{v}} \right] \quad (6)$$





Equation (6) must be same as equation (4). On comparing them



$$\left[ \frac{1 + \frac{v}{c}}{1 - \left( \frac{1}{\lambda^2} - 1 \right) \frac{c}{v}} \right] = 1 \quad (7)$$

And we get 
$$\lambda = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (8)$$

Putting the value of  $\lambda$  in equation (1) we get 
$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (9)$$

Similarly, on putting the value of  $x$  from eq<sup>n</sup> (6) and  $\lambda$  from eq<sup>n</sup> (8) into eq<sup>n</sup> (3) we get

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (10) \quad \text{So Lorentz transformation becomes}$$

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} & y' &= y \\ z' &= z & t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

Similarly, if  $S$  is moving with velocity  $v$  w.r.t.  $S'$  in opposite direction (-ve x-axis) then we get

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} & y &= y' \\ z &= z' & t &= \frac{t' + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

Inverse Lorentz Transformation

**Length Contraction:** In Newtonian mechanics, length of an object is same for all observers but according to STR length is a variable physical quantity and it contracts when the object moves.

Consider a rod of length  $L_0$  placed in a moving frame of reference  $S'$ . The observer  $O'$  observes the end coordinates of the rod as  $x_1'$  and  $x_2'$ .

Therefore, 
$$x_2' - x_1' = L_0 \quad (1)$$

And according to the observer  $O$  sitting in stationary frame of reference  $S$ , the coordinates of ends of the rod are  $x_1$  and  $x_2$ . Hence the observed length of the rod by the observer  $O$  is

$$L = x_2 - x_1 \quad (2)$$

Now, According to Lorentz transformation 
$$x_1' = \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad x_2' = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Put the value of  $x_1'$  and  $x_2'$  in eq<sup>n</sup> (1), we get

$$L_0 = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (3) \quad \text{or,} \quad \boxed{L = L_0 \sqrt{1 - \frac{v^2}{c^2}}} \quad (4) \quad \because \quad x_2 - x_1 = L$$

**NOTE:** From eq<sup>n</sup> (4), the observed length of the rod decreases as  $v$  increases.  
 Case 1: when  $(v/c) \ll 1$  then  $L=L_0$  (Length does not appear to be increased)  
 Case 2: when  $(v=c)$  then  $L=0$  (Length of the object contracts to a point)

**Time Dilation:** Time dilation is the lengthening of the time interval between two events for an observer in an inertial frame that is moving with respect to the rest frame of reference.

Consider two inertial frames  $S$  and  $S'$ .  $S'$  is moving with respect to  $S$  in +ve direction of x-axis with a velocity  $v$ . Imagine a gun placed at the fixed position  $(x', y', z')$  in the frame  $S'$ . Suppose it fires two shots at time  $t_1'$  and  $t_2'$  and observed by the observer  $O'$  sitting in  $S'$ .

According to observer  $O'$  the time interval between the shots is given by

$$t_2' - t_1' = t_0 \quad (\text{say}) \quad (1)$$

Now, according to observer  $O$  sitting in  $S$  the observed time interval between the shots is given by

$$t_2 - t_1 = t \quad (\text{say}) \quad (2)$$

From inverse Lorentz transformation equations, we have  $t_1 = \frac{t_1' + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$  (3) and  $t_2 = \frac{t_2' + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$  (4)

On putting the values of  $t_1$  and  $t_2$  in eq<sup>n</sup> (2), we have

$$t = \frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (5)$$

**NOTE:** Eq<sup>n</sup> (5) shows that  $t > t_0$  i.e. the time interval appears to be lengthened by a factor  $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$  which is observed by the observer in  $S$  frame. This is known as time dilation.

**Variation of mass with velocity:** In Newtonian physics, mass of a particle is considered to be a constant quantity. However, according to relativity the mass is also a variable quantity just like length and time.

The mass of a moving particle is given by

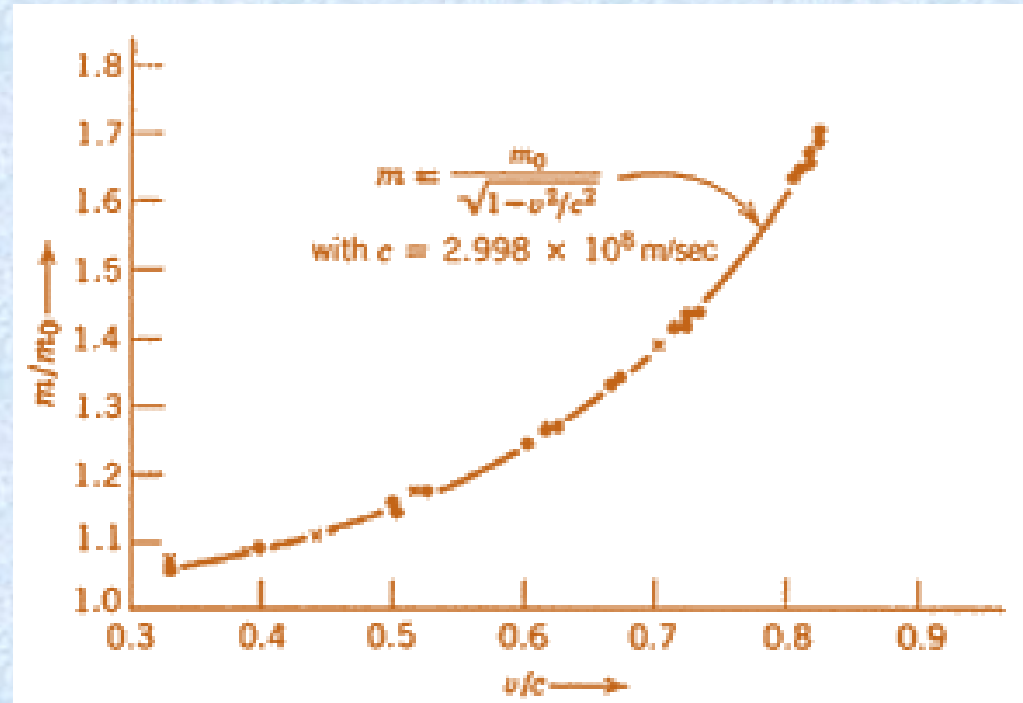
$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (1)$$

where,

$m_0$  = the rest mass of the particle (mass of the particle when it is at rest)

$v$  = velocity of the particle

$c$  = speed of light



**NOTE: Mass of the particle increases as the velocity increases**



# Einstein's Mass Energy Relation

According to STR, the mass of the moving particle is given by  $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$  (1)

If a force  $F$  is applied to the particle then it will be equal to the rate of change of momentum ( $mv$ ). Therefore,  $F = \frac{d(mv)}{dt}$  (2)

If the particle is displaced a distance  $dx$  by this force  $F$ , the work done  $F \cdot dx$  is stored as the kinetic energy ( $dE_k$ ). Hence  $dW = dE_k = F \cdot dx$  (3)

From eq<sup>n</sup> (2) and (3), we have  $dE_k = \frac{d(mv)}{dt} \cdot dx = \frac{dx}{dt} d(mv)$  (4)

or,  $dE_k = v d(mv) = v(mdv + vdm) \quad \because \frac{dx}{dt} = v \quad \Rightarrow \quad dE_k = (mv dv + v^2 dm)$  (5)

Using eq<sup>n</sup> (1), we have  $m^2 c^2 - m^2 v^2 = m_0^2 c^2$  (6)

Differentiating eq<sup>n</sup> (6) both side, we get

$$2m dm c^2 - 2m dm v^2 - 2v dv m^2 = 0 \quad \Rightarrow \quad c^2 dm = v^2 dm + mv dv \quad (8)$$

From eq<sup>n</sup> (5) and (8), we have  $dE_k = c^2 dm$  (9)

Integrate both sides  $\Rightarrow \int_0^{E_k} dE_k = c^2 \int_{m_0}^m dm \Rightarrow E_k = mc^2 - m_0 c^2$  (10)

Cont...

$$\therefore E = E_k + m_0c^2 \quad (11)$$

Total energy  
of the  
moving  
particle

Kinetic  
energy of  
the particle

Rest mass  
energy of  
the particle

From eq<sup>n</sup> (10) and (11), we have

$$\therefore E = mc^2 \quad (12)$$

Above eq<sup>n</sup> (12) is called mass energy relation

**Inference:** Energy of the particle can be converted into mass and vice-versa.

# Numerical Problems

1. Show that under Lorentz transformation  $x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$
2. With what velocity a particle should move so that its mass appears to increase by 20% of its rest mass?
3. If the kinetic energy of a body is doubled of its rest mass energy, calculate its velocity.
4. What is the length of a metre stick moving parallel to its length when its mass is  $3/2$  times of its rest mass?
5. Half life of a particle is 17.8 nanosecond. What will be the half life when its speed is  $0.8 c$  ?
6. With what velocity should a rocket move so that every year spent on it corresponds to 4 years on earth?