Mechanics and Wave Motion

(Paper-I)

For B.Sc. (I)

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Newtonian Mechanics and Conservation Laws

In Newtonian mechanics, Space, Time and Mass are regarded absolute i.e. Invariant

Newton's Laws of Motion:

First Law: A body continues in its state of rest or uniform motion in a straight line in the same direction unless some external force is applied to it. This law is also called *law of inertia*.

Second Law: The rate of change of linear momentum **(*p*=m*v*)** of a body is proportional to the force (***F***) applied and it takes place in a direction of the force, i.e.

$$\boldsymbol{F} \propto \frac{d\boldsymbol{p}}{dt} \qquad (1) \qquad \because \boldsymbol{p} = m\boldsymbol{v} \qquad \therefore \boldsymbol{F} = km\frac{d\boldsymbol{v}}{dt} \quad \Rightarrow \quad \boldsymbol{F} = km\boldsymbol{a}$$

where, k is a constant

Define force in such a way that k=1, we have $\boldsymbol{F} = m\,\boldsymbol{a}$ (2)

Third Law: To every action, there is equal and opposite reaction. Action and reaction act on different bodies.

Equations of Motion:

When a ***constant force*** is applied to a body then the equations of motion are a set of three equations that can be utilized to predict unknown information about an object's motion if other information is known.

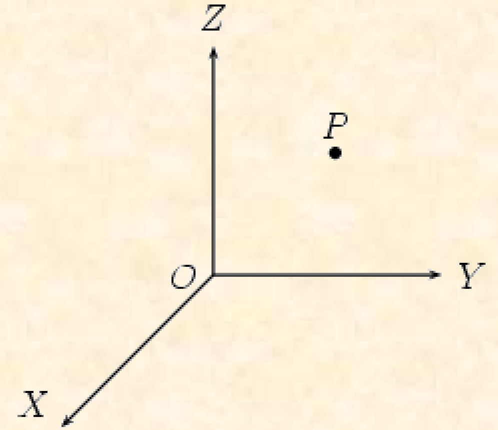
$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}\,t \qquad (1)$$

$$\boldsymbol{s} = \boldsymbol{u}\,t + \frac{1}{2}\boldsymbol{a}\,t^2 \qquad (2)$$

$$v^2 = u^2 + 2\,a\,s \qquad (3)$$

where, ***u***, ***v***, ***a*** and ***s*** are initial velocity, final velocity, acceleration and distance covered by body in time t respectively.

Frame of references: A system of coordinate axis, relative to which the motion of an object is described , is called a frame of reference.



Two types of frame of references:

1. **Inertial frame of reference:** Those frames of references, in which Newton's first and second laws hold true are called inertial frames.

or, Inertial frames of references are either at rest or move with constant velocity i.e. *Inertial frames as the frames with respect to which an unaccelerated body is unaccelerated*.

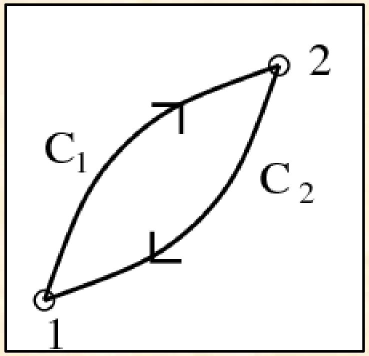
2. **Non Inertial frame of references:** Those frames of references in which Newton's first and second laws are not valid i.e. *Non inertial frames of references appear accelerated or decelerated with respect to an inertial frames of references.*

Concept of Fictitious Force: Consider a non inertial frame of reference S' moving with an acceleration $\boldsymbol{a_0}$ relative to an inertial frame of reference S. Then any particle of mass m which is at rest in S appears to move with an acceleration $-\boldsymbol{a_0}$ with respect to frame S'. Thus a force $-m\boldsymbol{a_0}$ will appear to be acting on the particle in frame S'. Such force which does not really act, but appears only due to the acceleration of the frame of reference is called *fictitious force* or *pseudo force*. The fictitious force is $\boldsymbol{F}=-m\boldsymbol{a_0}$.

If this mass m has an acceleration $\boldsymbol{a_i}$ in the inertial frame S, then the force observed in frame S' is given by

$$\boldsymbol{F}=m\boldsymbol{a_i} + (-m\boldsymbol{a_0})$$

Conservative Force: A force acting on a particle is said to be conservative if the work done by the force in moving a particle from one point to another is independent of the path.
or, *A force acting on a particle is said to be conservative if the kinetic energy of the particle remains unchanged for a complete round trip.*



In the figure, the work done by the conservative force from point 1 to point 2 along path C_1 is equal to the work done from point 2 to point 1 along path C_2.

Thus, the work done by a conservative force around closed path is zero.

The conservative force F is written as negative gradient of potential energy as given by

$$\boldsymbol{F} = -\nabla V \qquad \text{where,} \qquad \nabla = \boldsymbol{i}\frac{\partial}{dx} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

Here, **i, j, k** are unit vectors along x, y and z directions respectively

NOTE: Curl of a conservative force is always zero i.e. $\nabla \times \boldsymbol{F} = 0$

Principal of conservation of linear momentum: If the resultant force acting on the particle is zero, the linear momentum of the particle remains unaltered.

According to Newton's second law of motion

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \qquad \text{If F=0 then} \qquad \frac{d\boldsymbol{p}}{dt} = 0 \quad \Rightarrow \quad \boldsymbol{p} = m\boldsymbol{v} = \text{constant}$$

Centre of Mass: It is defined as the representative point which may describe the translatory motion of a body. Let the body consists of n particles of masses $m_1, m_2, m_3 \ldots m_n$, whose position vectors from any arbitrary origin are $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots\mathbf{r}_n$ respectively. The position of centre of mass $\mathbf{r}_{cm}$ is defined as

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3 + \ldots + m_n r_n}{m_1 + m_2 + m_3 + \ldots + m_n} \quad \Rightarrow \quad r_{cm} = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$

If there are large number of particles continuously distributed in space, then the coordinates of centre of mass are given by

$$x_{cm} = \frac{\rho}{M}\int x\, dV \qquad y_{cm} = \frac{\rho}{M}\int y\, dV \qquad z_{cm} = \frac{\rho}{M}\int z\, dV$$

where, ρ is the density of the whole body and *dV* is the infinitesimal volume element

Velocity of centre of mass:

$$\boldsymbol{v}_{cm} = \frac{d\boldsymbol{r}_{cm}}{dt} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{v}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \boldsymbol{p}_i}{M}$$

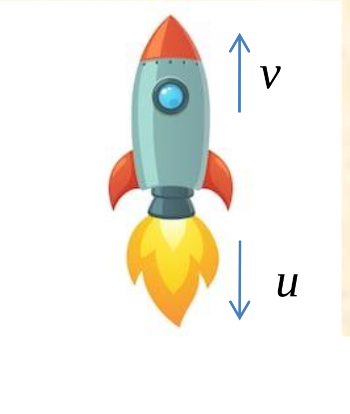
where, $\mathbf{v_i}$, and $\mathbf{p_i}$ are the velocity and momentum of i^{th} particle and M is the total mass of the body

Acceleration of centre of mass:

$$\boldsymbol{a}_{cm} = \frac{d\boldsymbol{v}_{cm}}{dt} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{a}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \boldsymbol{F}_i}{M}$$

where, $\mathbf{a_i}$, and $\mathbf{F_i}$ are the acceleration and Force corresponding to the i^{th} particle and M is the total mass of the body

Variable mass system (Rocket): Consider M_0 is the initial mass of the rocket and M is its mass after time t when its velocity relative to laboratory frame is v. Let α bet the rate at which mass of gasses leave the rocket with exhaust velocity of gases u (downward) relative to rocket.



As Mass of rocket-fuel system decreases due to burning of fuel therefore

$$\alpha = -\frac{dM}{dt} \qquad (1)$$

The momentum of rocket after time $t = M\,v$

$\therefore$ The rate of change of momentum of rocket $\left(\frac{dp}{dt}\right)_{rocket} = \frac{d}{dt}(Mv) = M\frac{dv}{dt} + v\frac{dM}{dt} \qquad (2)$

And, the rate of change of momentum of gasses is

$$\left(\frac{dp}{dt}\right)_{gases} = \alpha(v-u) = -\frac{dM}{dt}(v-u) \qquad (3)$$

At any time t,
v=velocity of rocket in laboratory frame (earth)
u= velocity of gases relative to rocket
v-u= velocity of gases in laboratory frame

Now, the rate of change of the system (rocket + gases) is given by

$$\left(\frac{dp}{dt}\right)_{rocket+gases} = \left(\frac{dp}{dt}\right)_{rocket} + \left(\frac{dp}{dt}\right)_{gasest} = M\frac{dv}{dt} + u\frac{dM}{dt} \qquad (4)$$

$\because \left(\frac{dp}{dt}\right)_{rocket+gases}$ = Resultant external force F acting on the system $\qquad \therefore \quad F = M\frac{dv}{dt} + u\frac{dM}{dt} \qquad (5)$

Case 1: When gravitational force is taken into account

$$F = -Mg \qquad (6)$$

From equation (5) and 6, we get $\quad dv + u\frac{dM}{M} = -g\,dt \qquad (7)$

On integrating both sides we get $\quad v + u\log_e M = -gt + C \qquad (8)$

where, C is a constant of integration and can be found by initial conditions

If initially, v=0 and M=M_0 then we have

$$C = u \log_e M_0 \qquad (9)$$

Now, from equation (8) and (9) $\quad v = u \log_e \frac{M_0}{M} - gt \qquad (10)$

Thus the effect of gravitational force decreases the speed of the rocket

Case 2. When rocket is out of gravitational field (g=0)

$$v = u \log_e \frac{M_0}{M} \qquad (11)$$

Remark: If at any instant rocket has some initial velocity $v_{0,}$ then after any time t its velocity will be

$$v = v_0 - u \log_e \frac{M}{M_0} - gt \qquad (12)$$

The above equation can be written as

$$v = v_0 - u \log_e \left(1 - \frac{\alpha t}{M_0}\right) - gt \qquad (13)$$

Need of Multistage Rocket: As we know the minimum velocity required for any body to escape the gravitational field of earth is 11.2 km/s which is called "Escape Velocity" of earth. Hence any rocket must have a velocity >11.2 km/s to escape the earth's gravitational field. Single stage rockets can not have this much velocity due to some limitations. Hence multistage rocket is required so that it can escape the earth's gravitational field.

From equation (11), for achieving a large final velocity *v*, it is necessary to have a large value of *u* and large value of M_0/M.
The value of *u* depends upon the temperature and pressure developed inside the combustion chamber.
The maximum value of *u* can be

$$u = v_{rms} = \sqrt{\frac{3RT}{M}} = 2 \text{ km/s}$$

where, M is the average molecular weight of the exhaust gases and T=3000°C (maximum temperature that can raise)

Also, in a single stage rocket the maximum value of M_0/M can reach around 10.

So the maximum velocity of a single stage rocket can be

$$v_I = u \log_e \frac{M_0}{M} = 2 \log_e 10 = 4.6 \text{ km/s} \qquad (14)$$

which is low as compared to Escape Velocity of earth. Hence more stages are required

For two stage rocket, the value of v_I of first stage will become *u* for second stage so the final velocity v_{II} of two stage rocket becomes

$$v_{II} = v_I + u \log_e \frac{M_0'}{M'} = 4.6 + 4.6 = 9.2 \text{ km/s} \qquad (15)$$

Also low value

Similarly, for three stage rocket, the final velocity v_{III} becomes

$$v_{II} = 9.2 + u \log_e \frac{M_0'}{M'} = 9.2 + 4.6 = 13.8 \text{ km/s} \qquad (16)$$

This value is greater than Escape Velocity of Earth. Hence at least three stages are required.

Rotational Dynamics: Moment of Inertia

Rigid Body: It is made up of a number of particles such that the relative distances of the constituent particles remain unaffected under the action of a force.
Two types of motion: (i) Translational (ii) Rotational

Moment of Inertia: The moment of inertia of a body for system of particles about an axis is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots = \sum_{i=1}^{n} m_i r_i^2$$

where, $m_1, m_2, m_3, \ldots m_n$ are masses of particles and $r_1, r_2, r_3, \ldots r_n$ their perpendicular distances from the axis of rotation respectively.

Angular momentum of a rigid body: The angular momentum (**J**) of a rigid body about the axis of rotation is equal to the product of moment of inertia (I) about that axis and angular velocity (**ω**).

$$\boldsymbol{J} = I\ \boldsymbol{\omega}$$

Torque (τ): About a fixed axis, it is defined as the product of moment of inertia and angular acceleration **(α).**

$$\tau = I\ \alpha$$

Kinetic energy of Rotation: Suppose we have a rigid body consists of *n* number of particles of masses $m_1, m_2, m_3, \ldots m_n$ and situated at positions $r_1, r_2, r_3, \ldots r_n$ from the axis of rotation. As the body rotates, all the particles will have same angular speeds (ω) but different linear speeds.

The kinetic energy of a particle of mass m_1 and linear speed v_1 is $E_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (r_1\omega)^2 \quad \because v_1 = r_1\omega$

The kinetic energy of a particle of mass m_2 and linear speed v_2 is $E_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 (r_2\omega)^2$

Hence, the total kinetic energy of the body is $E = E_1 + E_2 + E_3 + \ldots E_n$

$$E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots E \frac{1}{2} m_n r_n^2 \omega^2$$

$$E = \frac{1}{2} \omega^2 \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots m_n r_n^2 \right) \quad \Rightarrow \quad E = \frac{1}{2} I \omega^2$$

Total kinetic energy of a rolling body: If in addition to the rotational motion of the body of mass M, the body also moves along a certain path with a certain linear velocity *v* then the total kinetic energy of the body is given by

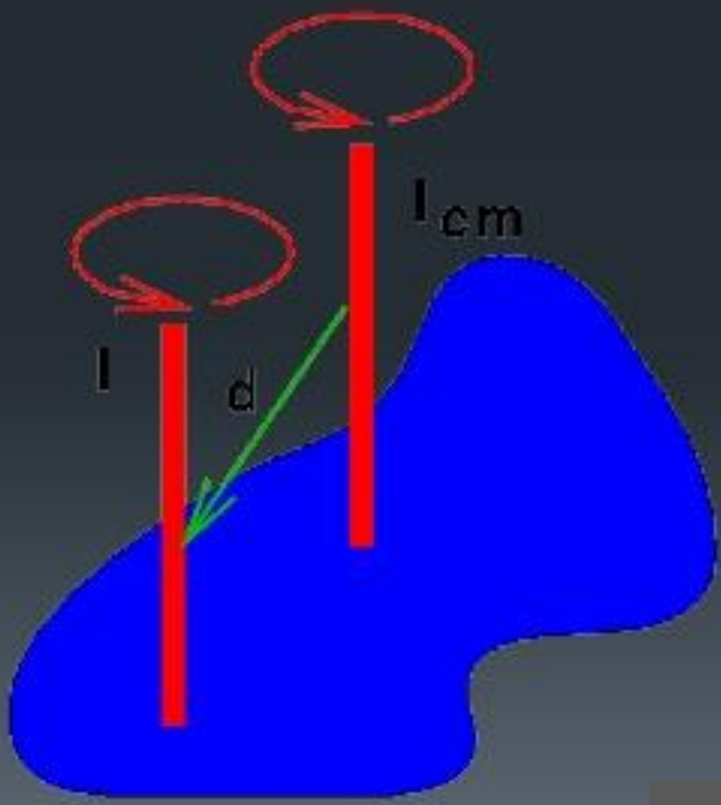
$$E_{Total} = \frac{1}{2} I \omega^2 + \frac{1}{2} M v^2$$

Theorem of Parallel Axes

Let I_{CM} be the moment of inertia of a body of mass M about an axis passing through the center of mass and let I be the moment of inertia about a parallel axis at a distance d from the first axis.

Then, $I = I_{CM} + Md^2$

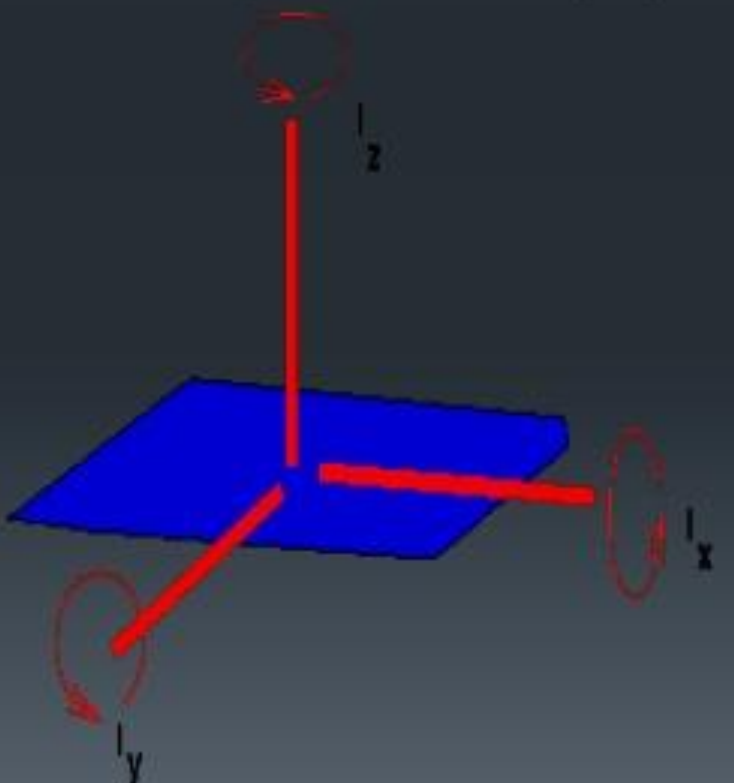
Thus the minimum moment of inertia for any object is at the center of mass, as x in the above expression is zero.

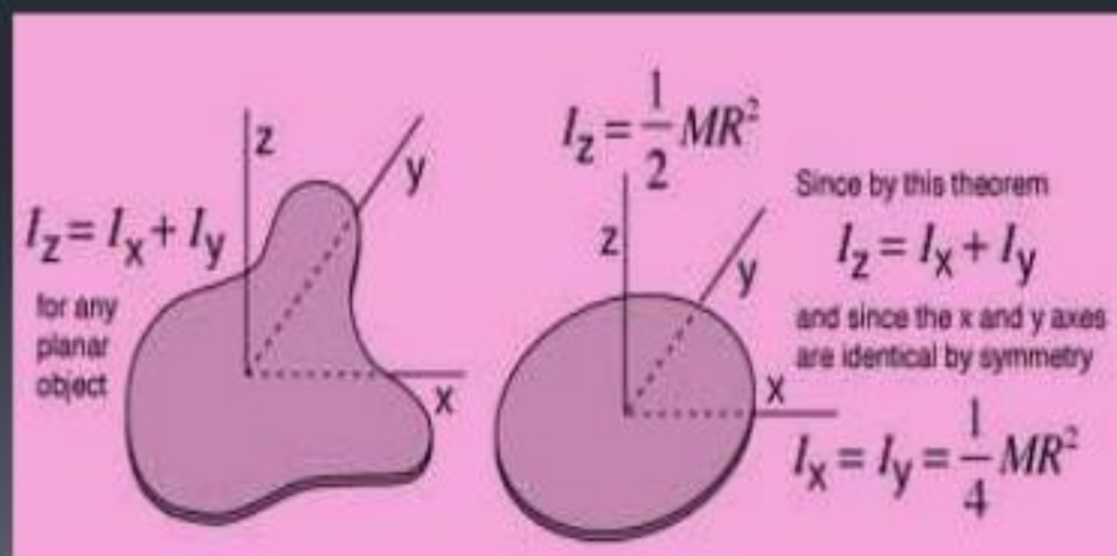


Theorem of perpendicular axis

The moment of inertia of a body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two mutually perpendicular axis in its own plane and crossing through the point through which the perpendicular axis passes

$$I_z = I_x + I_y$$



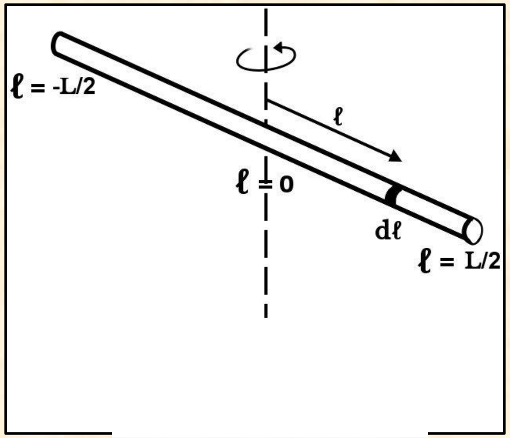


Moment of inertia of a thin uniform rod of mass M and length L

(i) About an axis passing through its centre and perpendicular to its length:

Consider, an element of length dl at a distance l from the centre. The mass of this element is dm=M.dl
Its moment of inertia about the axis is $dm.l^2$
The moment of inertia of the whole rod is given by

$$I_{cm} = \int_{l=-L/2}^{l=L/2} l^2 M\, dl = M \int_{l=-L/2}^{l=L/2} l^2 dl = \frac{M L^2}{12}$$

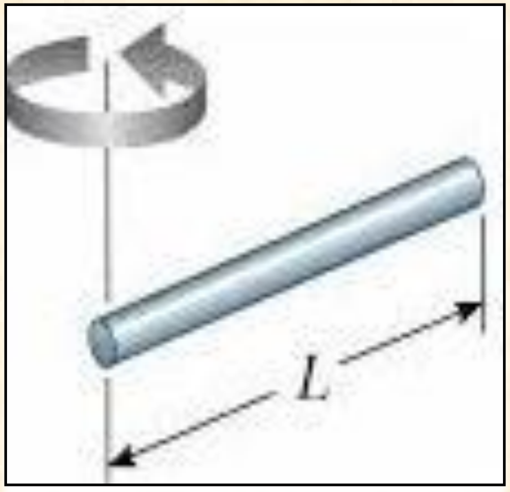


(ii) About an axis passing through its one end and perpendicular to its length:

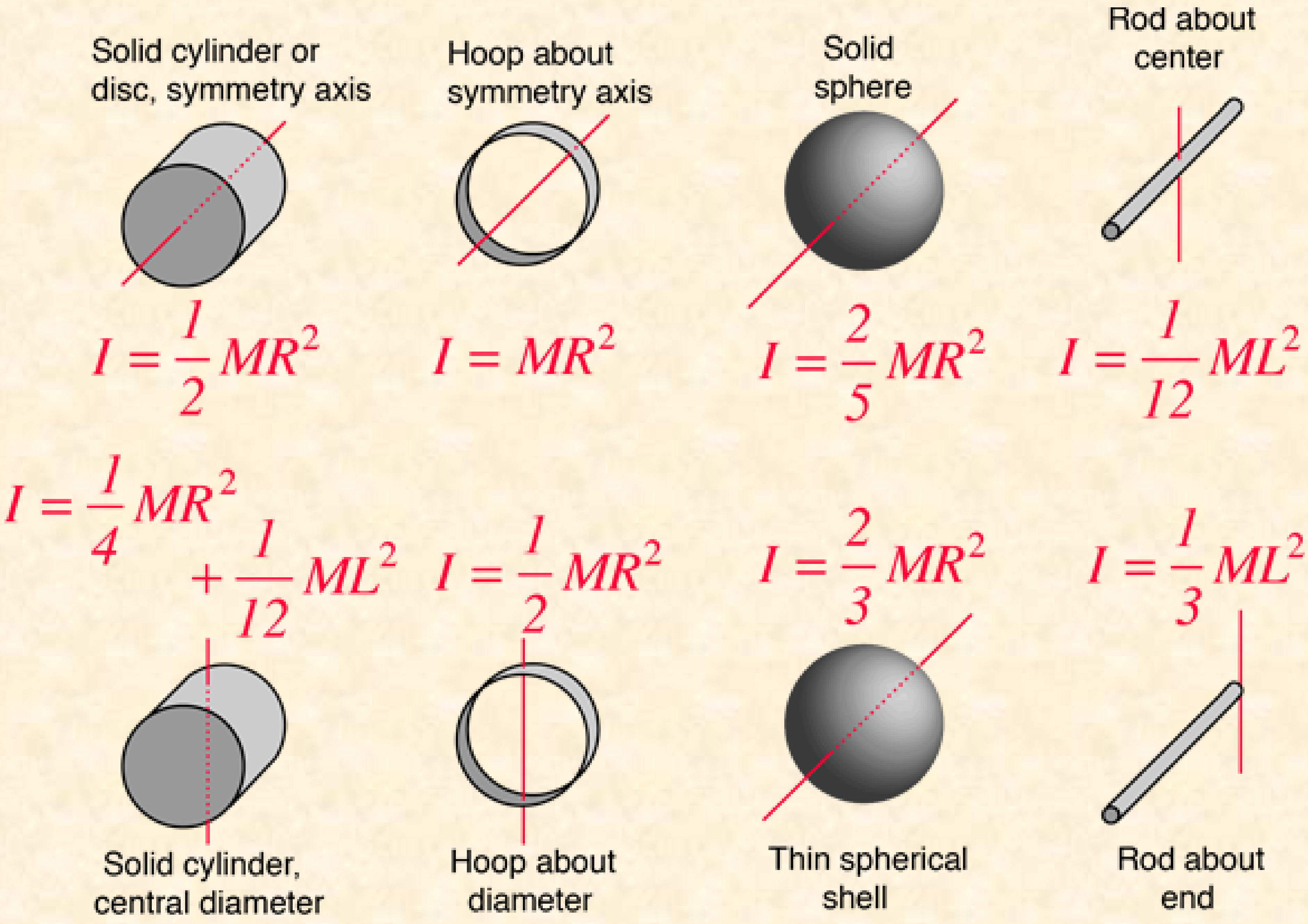
Use parallel axis theorem

$$I = I_{cm} + M\left(\frac{L}{2}\right)^2$$

$$I = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

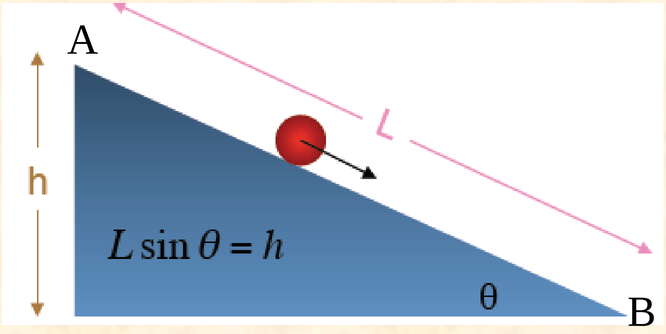


Moment of inertia (I) of some bodies about some axis of rotation:



Body rolling down an inclined plane (without slipping)

Consider a body shown in the fig. having mass M and radius R rolling freely (without slipping) down a plane inclined at an angle θ to the horizontal. In this case, the body has both rotational as well as translational motion. Let the body starts rolling from rest (point A) and when it reaches at point B its angular and linear velocities are ω and v respectively.



Apply conservation of energy at points A and B (*total energy at point A (E_A) must be equal to total energy at B (E_B)*)

$$E_A = M\,g\,L\sin\theta \qquad \text{(Only Potential Energy)} \qquad (1)$$

$$E_B = \frac{1}{2}M\,v^2 + \frac{1}{2}I\,\omega^2 \qquad [\text{Only Kinetic Energy (Translational + Rotational)}]$$

$$E_B = \frac{1}{2}M\,v^2 + \frac{1}{2}M\,k^2\frac{v^2}{R^2} \qquad \because I = M\,k^2 \quad \text{and} \quad \omega = \frac{v}{R} \qquad (2)$$

From (1) and (2) $\quad M\,g\,L\sin\theta = \frac{1}{2}M\,v^2 + \frac{1}{2}M\,k^2\frac{v^2}{R^2} \quad \Rightarrow \quad v^2 = 2L\,\dfrac{g\sin\theta}{1+\dfrac{k^2}{R^2}} \qquad (3)$

$$\because \quad v^2 = u^2 + 2aL \quad \Rightarrow \quad v^2 = 2aL \quad \text{(body starts from rest i.e. } u = 0) \qquad (4)$$

Comparing equations (3) and (4), we get

$$a = \frac{g\sin\theta}{1+\dfrac{k^2}{R^2}} \qquad (5)$$

Inference: From equation (5), greater the value of k as compared to R smaller will be the acceleration and hence greater the time taken by the body to reach at point B.

Numerical Problems

Example 1. Determine whether the Force ***F*** given below is conservative or not. Also compute the scalar potential (U) which generates it.

$$\boldsymbol{F} = 3x^2y^2\,\boldsymbol{i} + \left(2x^3y + \cos z\right)\boldsymbol{j} - y\ \sin z\,\boldsymbol{k}$$

Ans: ***F*** is conservative; $U = -\left(2x^3y + y\cos z\right)$

Example 2. If centre of masses of three particles of masses 1, 2 and 3 gm be at a point (1,1,2) then where should a fourth particle of mass 5 gm be placed so that the combined centre of mass may be at (0,0,0).

Ans: The position of fourth particle is given by $-\frac{6}{5}\left(\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\right)$

Example 3. Two masses of 6 and 2 units are at positions 6 i – 7 j and 2 i + 10 j -8 k respectively. Deduce the position of their center of mass.

Ans: 5 **i** – 2.75 **j** – 2 **k**

Example 4. A plane circular disc, a ring and a sphere of same radius are allowed to roll down an inclined plane. Explain which will reach the bottom last.

Example 5. Four spheres each of radius r and masses m are placed with their centres on four corners of a square of side l. Calculate the moment of inertia of the arrangement about any diagonal of the square.

Ans. $\frac{m}{5}\left(8r^2 + 5l^2\right)$