"Calculus"

(Jacobian)

For - B.Sc-I

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Definition. If $u_1, u_2, u_3, \ldots, u_n$ are functions of n independent variables $x_1, x_2, x_3, \ldots, x_n$ then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, u_3, \ldots, u_n$ with respect to $x_1, x_2, x_3, \ldots, x_n$ and is denoted either by $\frac{\partial(u_1, u_2, u_3, \ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)}$ OR by $J(u_1, u_2, u_3, \ldots, u_n)$.

Thus if u and v are functions of two independent variables x and y, then we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= J(u,v)$$

Similarly if u, v and w are functions of three independent variables x, y, and z then we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J(u,v,w)$$

For Ex. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = \cancel{r\cos\phi}$ $r\cos\theta$

show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Solution. we have

L.H.L $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta(r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi)$$
$$+ r\sin\theta(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi)$$

(expanding the determinante along the third row)

$$= r^2 \sin\theta \cos^2\theta + r^2 \sin^3\theta$$

$$= r^2 \sin\theta \quad \text{R.H.S}$$

⇒ Chain Rule [Case of functions of functions]

Theorem. If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2 then

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}$$

Proof- We have

$$\left.\begin{aligned}
\frac{\partial u_1}{\partial x_1} &= \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} \\
\frac{\partial u_1}{\partial x_2} &= \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} \\
\frac{\partial u_2}{\partial x_1} &= \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} \\
\frac{\partial u_2}{\partial x_2} &= \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2}
\end{aligned}\right\} \quad \text{——— (1)}$$

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Now. $\dfrac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \dfrac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \dfrac{\partial u_1}{\partial y_1} & \dfrac{\partial u_1}{\partial y_2} \\ \dfrac{\partial u_2}{\partial y_1} & \dfrac{\partial u_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial y_1} \cdot \dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial u_1}{\partial y_2} \cdot \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1} \cdot \dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial u_1}{\partial y_2} \cdot \dfrac{\partial y_2}{\partial x_2} \\ \dfrac{\partial u_2}{\partial y_1} \cdot \dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial u_2}{\partial y_2} \cdot \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_2}{\partial y_1} \dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial u_2}{\partial y_2} \cdot \dfrac{\partial y_2}{\partial x_2} \end{vmatrix}$$

by ~~using~~ using row-by-column multiplication rule

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}$$

$$\Rightarrow \boxed{\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}}$$

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Jacobian of Implicit functions

Theorem. If $u_1, u_2, u_3, \ldots, u_n$ instead of being given explicitly in terms of $x_1, x_2, x_3, \ldots, x_n$ are connected with them by equations such as

$$f_1(u_1, u_2, u_3, \ldots u_n, x_1, x_2, x_3, \ldots x_n) = 0$$

$$f_2(u_1, u_2, u_3, \ldots u_n, x_1, x_2, x_3, \ldots, x_n) = 0$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$f_n(u_1, u_2, u_3, \ldots, u_n, x_1, x_2, x_3, \ldots x_n) = 0$$

Then, we have
$$\frac{\partial(u_1, u_2, u_3, \ldots, u_n)}{\partial(x_1, x_2, x_3, \ldots, x_n)} = (-1)^n \frac{\dfrac{\partial(f_1, f_2, f_3, \ldots, f_n)}{\partial(x_1, x_2, x_3, \ldots x_n)}}{\dfrac{\partial(f_1, f_2, f_3, \ldots f_n)}{\partial(u_1, u_2, u_3, \ldots \ldots, u_n)}}$$

Proof. Here we shall find the result for two variables and the proof can be extended easily for n variables. Then we can write the proof for n variables on the basis of the proof given below for two variablees.

In the case of two variables, the connecting relations are

$$\left.\begin{aligned} f_1(u_1, u_2, x_1, x_2) &= 0 \\ f_1(u_1, u_2, x_1, x_2) &= 0 \end{aligned}\right] \text{———} \quad (1)$$

~~Expan~~ From equation (1), we have by differentiation

$$\left.\begin{aligned}
\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} &= 0 \\
\frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2} + \frac{\partial f_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2} &= 0 \\
\frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \frac{\partial f_2}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} &= 0 \\
\frac{\partial f_2}{\partial x_2} + \frac{\partial f_2}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2} + \frac{\partial f_2}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2} &= 0
\end{aligned}\right] \text{———} \quad (2)$$

Now $$\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)}\cdot\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} & \frac{\partial f_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2} + \frac{\partial f_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2} \\ \frac{\partial f_2}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \frac{\partial f_2}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} & \frac{\partial f_2}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2} + \frac{\partial f_2}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

by using row-by-column multiplication rule

$$= \begin{vmatrix} -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} \\ -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} \end{vmatrix}$$ from the relations (2)

$$= (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}$$

then we have $$\boxed{\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2 \frac{\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}}{\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)}}}$$

For Example

Prove that $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$

Solution - let $u = f_1(x, y)$, $v = f_2(x, y)$ —— (1)

obviously x and y can also be expressed as functions of u and v. Differentiating relations ① partially w.r.to u and v the we get

$$\left.\begin{array}{ll} 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, & 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, & 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{array}\right] - ②$$

Now $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

(by applying row-by-column multiplication

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$ from ②

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1}$$

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